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EIGENVALUES OF A LINEAR ELASTICA CARRYING LUMPED MASSES, SPRINGS AND VISCOUS DAMPERS

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1. INTRODUCTION

The free vibration of an undamped linear elastic structure carrying any number of lumped springs and masses has received considerable interest in recent years, and has been studied by many authors. The most commonly used approaches to obtain the natural frequencies of such systems include the Lagrange multipliers approach [1–3], the dynamic Green function scheme [4–6], and the assumed modes method [7–9]. In all of the systems considered in references [1–9], no damping was present. Recently, Gürgöze and Erol [10] investigated the eigenvalues of a longitudinally vibrating rod carrying tip mass and viscously damped spring–mass in-span. They first solved the frequency equation exactly, and then used the Lagrange's equations in conjunction with the Lagrange multipliers formalism to obtain the approximate frequency equation. Although the results are concise, the inherent natural of the Lagrange multipliers formalism misses certain eigenvalues when the damped oscillator is located at a node of any normal mode of the rod. In addition, the Lagrange multipliers approach can be fairly laborious to apply, because S Lagrange multipliers need to be introduced and additional S constraint equations need to be formulated, where S is the number of distinct attachment locations. More recently, Chang *et al.* [11] utilized the Laplace transform with respect to the spatial variable to analyze the free vibration of a simple beam carrying point masses, grounded springs and grounded viscous dampers. While conceptually simple, the approach is rather tedious to apply because one needs to perform an inverse Laplace transform and enforce the boundary conditions to obtain the characteristic equation, the steps of which can be very algebraically intensive. In addition, the resulting characteristic equation is complicated, lengthy and difficult to code.

In this technical note, the discretized governing equations for a linear elastica carrying a number of lumped masses, springs and viscous dampers (see Figure 1) are first obtained by using the common assumed-modes method. Manipulating the characteristic determinant that governs the eigenvalues of the system, the characteristic determinant can be algebraically reduced to one of a smaller size, thus providing an alternative means to solve for the eigenvalues of the combined structure. The advantages of the proposed scheme will be discussed and highlighted. The utility of the proposed technique will be demonstrated by considering various example problems, and the results will be compared to known numerical and analytical solutions.

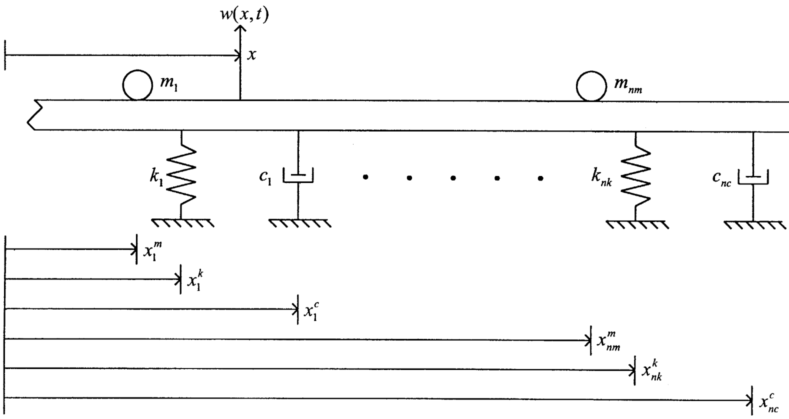

Figure 1. An arbitrarily supported, linear elastic structure carrying any number of lumped masses, grounded springs and grounded viscous dampers.

2. THEORY

Consider the free vibration of an arbitrarily supported, linear elastic structure carrying *nm* lumped masses, *nk* grounded translational springs and *nc* grounded viscous dampers as shown in Figure 1, where all the lumped elements are assumed to be attached at distinct locations. Using the assumed-modes method [12], the physical deflection of the structure at a point x is given by

$$w(x, t) = \sum_{i=1}^{N} \phi_i(x)\eta_i(t), \tag{1}$$

where $\phi_i(x)$ are the eigenfunctions of the linear structure (the elastica without any lumped attachments) that serve as the basis functions for this approximate solution, $\eta_i(t)$ are the corresponding generalized co-ordinates, and N is the number of modes used in the assumed-modes expansion. The total kinetic energy of the combined system is given by

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}\sum_{i=1}^{nm} m_i\dot{w}^2(x_i^m, t), \tag{2}$$

where M_i are the generalized masses of the linear elastica, m_i is the ith lumped mass, x_i^m is its location, and an overdot denotes a time derivative. The total potential energy is given by

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}\sum_{i=1}^{nk} k_i w^2(x_i^k, t), \tag{3}$$

where K_i are the generalized spring constants of the linear elastica, k_i is the stiffness of the ith grounded spring, and x_i^k represents its location. The Rayleigh's dissipation function for the combined structure is

$$R = \frac{1}{2}\sum_{i=1}^{nc} c_i\dot{w}^2(x_i^c, t), \tag{4}$$

where c_i is the ith viscous damping coefficient and x_i^c is its location.

Substituting equation (1) into equations (2)–(4) and applying Lagrange's equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\eta}_i}\right) - \frac{\partial T}{\partial \eta_i} + \frac{\partial V}{\partial \eta_i} + \frac{\partial R}{\partial \dot{\eta}_i} = 0, \quad i = 1, 2, \ldots, N \tag{5}$$

the equations of motion for the system of Figure 1 can be readily obtained. After some algebra, they are found to be governed by the homogeneous matrix equation

$$[M]\underline{\ddot{\eta}}(t) + [C]\underline{\ddot{\eta}}(t) + [K]\underline{\eta}(t) = \underline{0}, \tag{6}$$

where $\underline{\eta}(t) = [\eta_1(t) \;\; \eta_2(t) \;\; \ldots \;\; \eta_N(t)]^{\mathrm{T}}$ is the vector of generalized co-ordinates, and

$$[M] = [M^d] + \sum_{i=1}^{nm} m_i \underline{\phi}(x_i^m)\underline{\phi}^{\mathrm{T}}(x_i^m), \tag{7}$$

$$[K] = [K^d] + \sum_{i=1}^{nk} k_i \underline{\phi}(x_i^k)\underline{\phi}^{\mathrm{T}}(x_i^k), \quad [C] = \sum_{i=1}^{nc} c_i \underline{\phi}(x_i^c)\underline{\phi}^{\mathrm{T}}(x_i^c). \tag{8, 9}$$

Matrices $[M^d]$ and $[K^d]$ are both diagonal, whose elements are the generalized masses and stiffnesses, M_i and K_i respectively. It should be noted that both $[M]$ and $[K]$ consist of a diagonal matrix modified by nm and nk rank one matrices, respectively, and that $[C]$ consists of the sum of nc rank one matrices. Finally, all three system matrices are symmetric and of size $N \times N$.

Because equation (6) represents a homogeneous set of ordinary differential equations with constant coefficients, its solution has the exponential form

$$\underline{\eta}(t) = \underline{\bar{\eta}}\,\mathrm{e}^{\lambda t}, \tag{10}$$

where λ is a constant scalar and $\underline{\bar{\eta}}$ is a constant vector. Inserting equation (10) into equation (6) leads to

$$\{\lambda^2[M] + [C]\lambda + [K]\}\underline{\bar{\eta}}\,\mathrm{e}^{\lambda t} = \underline{0}. \tag{11}$$

Because an exponential can never be zero, in order to have a non-trivial solution for $\underline{\bar{\eta}}$, the exponent λ must satisfy the following characteristic determinant:

$$\det\{\lambda^2[M] + [C]\lambda + [K]\} = 0. \tag{12}$$

Expanding equation (12) leads to a $2N$ order polynomial in λ. Once the coefficients, the α_i's, of the polynomial are found and properly stored in a vector $\underline{\alpha}$, where the $2N + 1$ coefficients are arranged such that α_1 to α_{2N+1} correspond to the coefficients of the highest to the lowest power in λ, then the λ's can be readily solved using any prepackaged code such as *rpzero* in CMLIB [13] or *roots* in MATLAB. Unfortunately, while conceptually simple, to the best knowledge of the present author, there is no code that expands equation (12) directly, and constructs the properly arranged vector of coefficients, $\underline{\alpha}$.

The constant scalar λ can also be obtained by using a state-space approach, which effectively replaces N coupled second order differential equations by $2N$ coupled first order ordinary differential equations as follows [12]. A state vector of length $2N$ is introduced,

$$\underline{q}(t) = [\underline{\eta}^{\mathrm{T}}(t) \quad \underline{\dot{\eta}}^{\mathrm{T}}(t)]^{\mathrm{T}}, \tag{13}$$

such that equation (6) can be rewritten in a form that consists of $2N$ simultaneous first order ordinary differential equations as

$$[A]\underline{\dot{q}}(t) - [B]\underline{q}(t) = \underline{0}, \tag{14}$$

where matrices $[A]$ and $[B]$ are both symmetric and are given by

$$[A] = \begin{bmatrix} [0] & [M] \\ [M] & [C] \end{bmatrix} \quad \text{and} \quad [B] = \begin{bmatrix} [M] & [0] \\ [0] & -[K] \end{bmatrix}. \tag{15}$$

Because equation (14) is homogeneous, its solution is given by

$$\underline{q}(t) = \underline{\bar{q}}\,\mathrm{e}^{\lambda t}. \tag{16}$$

Substituting equation (16) into equation (14) yields the $2N \times 2N$ generalized eigenvalue problem

$$[B]\underline{\bar{q}} = \lambda[A]\underline{\bar{q}}, \tag{17}$$

where λ corresponds to the eigenvalue of the system. Equation (17) can be readily solved by using any existing prepackaged code such as *rsg* in EISPACK or *eig* in MATLAB.

In this technical note, an alternative approach is introduced to obtain the eigenvalues for the system of Figure 1. Instead of expanding equation (12) and then solving for the roots or eigenvalues, the λ's, of the resulting $2N$ order polynomial, equation (12) can be manipulated into a form that is immediately amenable to the solution scheme introduced in reference [14]. Substituting equations (7)–(9) into equation (12) leads to

$$\det\left\{[K^d] + \lambda^2[M^d] + \sum_{i=1}^{n} \sigma_i \underline{\phi}(x_i)\underline{\phi}^{\mathrm{T}}(x_i)\right\} = 0, \tag{18}$$

where $n = nm + ns + nc$, and

$$\sigma_i = m_i\lambda^2, \quad x_i = x_i^m, \quad i = 1, \ldots, nm, \tag{19}$$

$$\sigma_{i+nm} = k_i, \quad x_{i+nm} = x_i^k, \quad i = 1, \ldots, nk, \tag{20}$$

$$\sigma_{i+nm+nk} = c_i\lambda, \quad x_{i+nm+nk} = x_i^c, \quad i = 1, \ldots, nc. \tag{21}$$

Expanding equation (18), we have

$$\det\{[K^d] + \lambda^2[M^d]\}\det\left\{[I] + \sum_{i=1}^{n} \sigma_i([K^d] + \lambda^2[M^d])^{-1}\underline{\phi}(x_i)\underline{\phi}^{\mathrm{T}}(x_i)\right\} = 0, \tag{22}$$

which can be shown [14] to be identical to

$$\det\{[K^d] + \lambda^2[M^d]\}\det[B] = \left\{\prod_{i=1}^{N}(K_i + \lambda^2 M_i)\right\}\det[B] = f(\lambda) = 0, \tag{23}$$

where the K_i and M_i represent the ith element of $[K^d]$ and $[M^d]$, and the (i, j)th element of $[B]$, of size $n \times n$ is given by

$$b_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi_r(x_j)}{K_r + \lambda^2 M_r} + \frac{1}{\sigma_i}\,\delta_i^j, \quad i, j = 1, \ldots, n \tag{24}$$

and δ_i^j represents the Kronecker delta. The product terms of equation (23) are significant because they serve as a reminder that when the attachment locations for the lumped elements coincide with the nodes of any component mode, $\phi_i(x)$, then some of the eigenvalues, the λ's, of the combined system will be identical to j, the complex unity, times the natural frequencies of the linear structure (the elastica without any lumped attachments). Physically, this simply means that when the attachment locations coincide with the nodes of any component mode, some natural frequencies of the combined system will be identical to those of the linear structure.

When the system is undamped, i.e., $c_i = 0$, then the eigenvalues λ are purely imaginary, implying that the system executes simple harmonic motion, consistent with physical intuition. In this case, $\lambda = \mathrm{j}\omega$, where ω represents the undamped natural frequency of the system, and equations (23) and (24) reduce to

$$\det\{[K^d] - \omega^2[M^d]\}\det[B] = \left\{\prod_{i=1}^{N}(K_i - \omega^2 M_i)\right\}\det[B] = f(\omega) = 0 \tag{25}$$

and

$$b_{ij} = \sum_{r=1}^{N} \frac{\phi_r(x_i)\phi_r(x_j)}{K_r - \omega^2 M_r} + \frac{1}{\sigma_i}\,\delta_i^j, \quad i, j = 1, \ldots, nm + nk \tag{26}$$

the same results as equations (8) and (9) of reference [14]. The natural frequencies, ω, can be readily obtained graphically by plotting $f(\omega)$ as a function of ω and locating the zeros of $f(\omega)$. They can also be solved numerically using any existing subroutine such as *zeroin* in CMLIB or *fzero* in MATLAB.

When damping is present, the problem becomes more interesting, because the eigenvalues may now be complex. Thus, a complex solution of the form

$$\lambda = \lambda_r + \mathrm{j}\lambda_i \tag{27}$$

is assumed from the outset, where λ_r and λ_i correspond to the real and imaginary parts of λ respectively. In this case, the eigenvalues, λ, can be obtained graphically by examining the surface plot of $|f(\lambda)|$, i.e., the magnitude of the complex function $f(\lambda)$, as a function of λ_r and λ_i. The zeros of the surface plot correspond to the solution of $f(\lambda) = 0$. Alternatively, the eigenvalues can also be computed numerically. Substituting equation (27) into equations (23) and (24), setting the real and imaginary parts of the resulting function equal to zero, and solving them simultaneously using existing codes such as *snsqe* in CMLIB or *fsolve* in MATLAB, the complex eigenvalues can be readily obtained.

The proposed scheme of determining the eigenvalues, the λ's, of a linear elastica carrying any number of lumped masses, grounded springs and grounded viscous dampers offers numerous advantages. Firstly, equation (23) is simple to code and compact. Given the eigenfunctions, $\phi_i(x)$, of the linear elastica, the parameters for the masses, springs and dampers, m_i, k_i and c_i, and the attachment locations, x_i^m, x_i^k and x_i^c, equation (23) can be easily programmed. Moreover, the resulting characteristic determinant (see equation (23)) is substantially more concise compared to equation (8) of reference [11]. Secondly, the proposed method lends itself easily to a graphical means of solving for the λ's. Thirdly, the λ's can also be easily solved numerically by using any existing prepackaged code. Finally, the approach outlined in this note can be extended to accommodate any linear elastic structure with any arbitrary boundary conditions by simply using the appropriate eigenfunctions.

3. RESULTS

To show the utility of the proposed scheme, the eigenvalues of a uniform simply supported Euler–Bernoulli beam carrying any number of lumped elements are computed, and the results are compared to published values in reference [11], where the system parameters are as follows: $\rho = 1{\cdot}6363 \times 10^4$ kg/m (mass per unit length); $E = 2{\cdot}756 \times 10^{10}$ N/m^2 (Young's modulus); $L = 15{\cdot}24$ m (length of beam); and $I = 6{\cdot}0482\,\mathrm{m}^4$ (area moment of inertia of the cross-section). In all of the subsequent numerical examples, $N = 40$ (the number of component modes used in the assumed modes method), and because MATLAB is easy to code, MATLAB routines are used to solve for the eigenvalues. Specifically, when the system is undamped, MATLAB routine *fzero* is used and when the system is damped, MATLAB routine *fsolve* is called. For a simply supported beam, its normalized (with respect to ρ) eigenfunctions are given by

$$\phi_i(x) = \sqrt{\frac{2}{\rho L}} \sin \frac{\mathrm{i}\pi x}{L} \tag{28}$$

such that the generalized masses and stiffnesses of the beam become

$$M_i = 1 \quad \text{and} \quad K_i = (\mathrm{i}\pi)^4 EI/(\rho L^4). \tag{29}$$

Consider first the case of a uniform simply supported Euler–Bernoulli beam carrying one concentrated mass, m_1, at x_1^m, in which case equation (25) reduces to the simple frequency equation

$$\prod_{i=1}^{N} (K_i - \omega^2 M_i)\left(1 - m_1\omega^2 \sum_{r=1}^{N} \frac{\phi_i^2(x_1^m)}{K_i - \omega^2 M_i}\right) = 0. \tag{30}$$

Table 1 compares the first three natural frequencies obtained by solving equation (30) and those given in reference [11]. Since the attachment location, $x_1^m = 0{\cdot}5L$, coincides with the

TABLE 1

The first three natural frequencies of a simply supported, uniform Euler–Bernoulli beam carrying a mass, $m_1 = 0{\cdot}1\rho L$, at $0{\cdot}5L$

ω (rad/s)	Reference [11]	Present ($N = 40$)
ω_1	1·237859e + 002	1·237859e + 002
ω_2	5·425147e + 002	5·425144e + 002
ω_3	1·127885e + 003	1·127887e + 003

TABLE 2

The first three natural frequencies of a simply supported, uniform Euler–Bernoulli beam carrying a grounded spring, $k_1 = 0{\cdot}1\pi^2 EI/L^3$, at $0{\cdot}5L$

ω (rad/s)	Reference [11]	Present ($N = 40$)
ω_1	1·485370e + 002	1·485370e + 002
ω_2	5·425147e + 002	5·425144e + 002
ω_3	1·222167e + 003	1·222166e + 003

TABLE 3

The first three natural frequencies of a simply supported, uniform Euler–Bernoulli beam carrying a grounded damper, $c_1 = 0{\cdot}3\pi^2\sqrt{EI\rho/L^2}$, at $0{\cdot}5L$

λ (rad/s)	Reference [11]	Present ($N = 40$)
λ_1	− 4·090648e + 001 ± 1·296947e + 002j	− 4·090650e + 001 ± 1·296946e + 002j
λ_2	−1·331735e − 012 ± 5·425147e + 002j	0 ± 5·425144e + 002j
λ_3	− 4·061522e + 001 ± 1·217792e + 003j	− 4·061524e + 001 ± 1·217791e + 003j

node of the even component modes of a uniform simply supported beam, equation (30) reminds us (see the product terms) that all the even natural frequencies of the combined system are identical to those of the simply supported beam. Thus, $\omega_2^2 = (2\pi)^4 EI/(\rho L^4)$. From Table 1, note the excellent agreement between the results of equation (30) and the solution given in reference [11].

Consider now the case where the simply supported beam is carrying a grounded spring, of stiffness k_1, at $x_1^k = 0{\cdot}5L$, in which case equation (25) simplifies to

$$\prod_{i=1}^{N}(K_i - \omega^2 M_i)\left(1 + k_1 \sum_{r=1}^{N} \frac{\phi_i^2(x_1^k)}{K_i - \omega^2 M_i}\right) = 0. \tag{31}$$

Table 2 compares the results given in reference [11] and those obtained by solving equation (31). Note the excellent agreement between the two solutions. Because the attachment point is at the midspan, as expected, the second natural frequency of the combined system coincides with the second natural frequency of the simply supported Euler–Bernoulli beam.

Next, consider the case where a grounded viscous damper, of damping coefficient c_1, is attached to a uniform simply supported Euler–Bernoulli beam at $x_1^c = 0{\cdot}5L$. The eigenvalues of this particular system is given by

$$\prod_{i=1}^{N}(K_i + \lambda^2 M_i)\left(1 + c_1\lambda \sum_{r=1}^{N} \frac{\phi_i^2(x_1^c)}{K_i + \lambda^2 M_i}\right) = 0. \tag{32}$$

Because there is damping in the system, the eigenvalues will necessarily be complex. Table 3 compares the first three eigenvalues obtained by solving equation (32) and those given in reference [11]. Note again how well the present results track those listed in reference [11].

Table 4 shows the first three eigenvalues of a uniform simply supported beam, to which one lumped mass, m_1, one grounded spring, k_1, and one grounded viscous damper, c_1, are

TABLE 4

The first three natural frequencies of a simply supported, uniform Euler–Bernoulli beam carrying a concentrated mass, $m_1 = 0{\cdot}1\rho L$, a grounded spring, $k_1 = 0{\cdot}1\pi^2 EI/L^3$, and a grounded damper, $c_1 = 0{\cdot}1\pi^2\sqrt{EI\rho/L^2}$, all at $0{\cdot}5L$

λ (rad/s)	Reference [11]	Present ($N = 40$)
λ_1	$-1{\cdot}130626\text{e}+001 \pm 1{\cdot}351799\text{e}+002\text{j}$	$-1{\cdot}130627\text{e}+001 \pm 1{\cdot}351799\text{e}+002\text{j}$
λ_2	$-4{\cdot}177324\text{e}-012 \pm 5{\cdot}425147\text{e}+002\text{j}$	$0 \pm 5{\cdot}425144\text{e}+002\text{j}$
λ_3	$-8{\cdot}482803\text{e}+000 \pm 1{\cdot}128716\text{e}+003\text{j}$	$-8{\cdot}482364\text{e}+000 \pm 1{\cdot}128718\text{e}+003\text{j}$

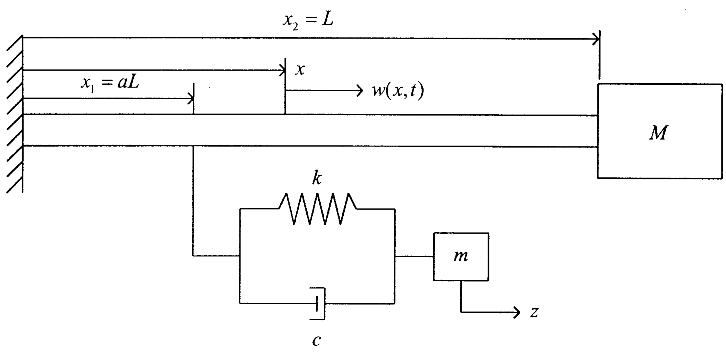

Figure 2. A fixed-free, longitudinally vibrating elastica carrying a tip mass and a viscously damped oscillator in-span.

attached at $0{\cdot}5L$. For this particular case, the complex eigenvalues are obtained by solving equation (23). From Table 4, note the excellent agreement between the results of equation (23) and those given in reference [11].

To further illustrate the versatility of the proposed approach, consider the system of Figure 2, which consists of a linear elastica carrying a tip mass and viscously damped spring–mass in-span. This particular system was analyzed by Gürgöze and Erol [10], who used the Lagrange multipliers formalism to obtain the frequency equation of the system. Here, it will be shown that the proposed approach can be extended to obtain the same frequency equation with simple algebraic manipulations. For the system of Figure 2, the total kinetic energy, the total potential energy and the Rayleigh's dissipation functions are given by

$$T = \frac{1}{2}\sum_{i=1}^{N} M_i\dot{\eta}_i^2(t) + \frac{1}{2}m\dot{z} + \frac{1}{2}M\dot{w}^2(L, t), \tag{33}$$

$$V = \frac{1}{2}\sum_{i=1}^{N} K_i\eta_i^2(t) + \frac{1}{2}k[w(aL, t) - z(t)]^2, \qquad R = \frac{1}{2}c[\dot{w}(aL, t) - \dot{z}(t)]^2. \tag{34, 35}$$

Applying Lagrange's equations, the following equations of motion are obtained:

$$\begin{bmatrix} [\mathscr{M}] & \underline{0} \\ \underline{0}^{\mathrm{T}} & m \end{bmatrix}\begin{bmatrix} \ddot{\underline{\eta}} \\ \ddot{z} \end{bmatrix} + \begin{bmatrix} [\mathscr{C}] & -c\underline{\phi}_a \\ -c\underline{\phi}_a^{\mathrm{T}} & c \end{bmatrix}\begin{bmatrix} \dot{\underline{\eta}} \\ \dot{z} \end{bmatrix} + \begin{bmatrix} [\mathscr{K}] & -k\underline{\phi}_a \\ -k\underline{\phi}_a^{\mathrm{T}} & k \end{bmatrix}\begin{bmatrix} \underline{\eta} \\ z \end{bmatrix} = \begin{bmatrix} \underline{0} \\ 0 \end{bmatrix}, \tag{36}$$

where $\underline{\phi}_a$ corresponds to $\underline{\phi}(x)$ evaluated at $x = aL$, and

$$[\mathscr{M}] = [M^d] + M\underline{\phi}(L)\underline{\phi}^{\mathrm{T}}(L) \tag{37}$$

$$[\mathscr{K}] = [K^d] + k\underline{\phi}(aL)\underline{\phi}^{\mathrm{T}}(aL), \qquad [\mathscr{C}] = c\underline{\phi}(aL)\underline{\phi}^{\mathrm{T}}(aL). \tag{38, 39}$$

Assuming an exponential solution,

$$\begin{bmatrix} \underline{\eta} \\ z \end{bmatrix} = \begin{bmatrix} \bar{\underline{\eta}} \\ \bar{z} \end{bmatrix} \mathrm{e}^{\lambda t}, \tag{40}$$

equation (36) becomes

$$\left\{ \lambda^2 \begin{bmatrix} [\mathscr{M}] & \underline{0} \\ \underline{0}^{\mathrm{T}} & m \end{bmatrix} + \lambda \begin{bmatrix} [\mathscr{C}] & -c\underline{\phi}_a \\ -c\underline{\phi}_a^{\mathrm{T}} & c \end{bmatrix} + \begin{bmatrix} [\mathscr{K}] & -k\underline{\phi}_a \\ -k\underline{\phi}_a^{\mathrm{T}} & k \end{bmatrix} \right\} \begin{bmatrix} \bar{\underline{\eta}} \\ \bar{z} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ 0 \end{bmatrix}. \tag{41}$$

Using the last equation of equation (41) to obtain an expression for $\bar{z}$ in terms of $\bar{\underline{\eta}}$ yields

$$\bar{z} = \frac{c\lambda + k}{\lambda^2 m + c\lambda + k} \underline{\phi}_a^{\mathrm{T}} \bar{\underline{\eta}}. \tag{42}$$

Substituting the above into the top equation of equation (41) leads to

$$\left\{ \lambda^2[\mathscr{M}] + \lambda[\mathscr{C}] + [\mathscr{K}] - \frac{(c\lambda + k)^2}{\lambda^2 m + \lambda c + k} \underline{\phi}_a \underline{\phi}_a^{\mathrm{T}} \right\} \bar{\underline{\eta}} = \underline{0}. \tag{43}$$

After some algebra, equation (43) reduces to

$$\left\{ \lambda^2[M^d] + \lambda^2 M\underline{\phi}(L)\underline{\phi}^{\mathrm{T}}(L) + [K^d] + \frac{(c\lambda + k)m\lambda^2}{\lambda^2 m + \lambda c + k} \underline{\phi}(aL)\underline{\phi}^{\mathrm{T}}(aL) \right\} \bar{\underline{\eta}} = \underline{0}. \tag{44}$$

For a non-trivial solution, the exponent λ must satisfy

$$\det \left\{ [K^d] + \lambda^2[M^d] + \sum_{i=1}^{2} \sigma_i \underline{\phi}(x_i)\underline{\phi}^{\mathrm{T}}(x_i) \right\} = 0, \tag{45}$$

where $x_1 = aL$, $x_2 = L$, and

$$\sigma_1 = \frac{(c\lambda + k)m\lambda^2}{m\lambda^2 + c\lambda + k}, \qquad \sigma_2 = \lambda^2 M. \tag{46}$$

Note that equation (45) has the same form as equation (18). Using the results derived previously, the characteristic determinant of equation (45) can be immediately reduced to

$$\left\{ \prod_{i=1}^{N} (K_i + \lambda^2 M_i) \right\} \det \begin{bmatrix} \sum_{r=1}^{N} \frac{\phi_r^2(x_1)}{K_r + \lambda^2 M_r} + \frac{1}{\sigma_1} & \sum_{r=1}^{N} \frac{\phi_r(x_1)\phi_r(x_2)}{K_r + \lambda^2 M_r} \\ \sum_{r=1}^{N} \frac{\phi_r(x_2)\phi_r(x_1)}{K_r + \lambda^2 M_r} & \sum_{r=1}^{N} \frac{\phi_r^2(x_2)}{K_r + \lambda^2 M_r} + \frac{1}{\sigma_2} \end{bmatrix} = 0. \tag{47}$$

When the eigenfunctions, $\phi_r(x)$, are properly normalized with respect to the mass per unit length, ρ, of the linear elastica, the generalized masses and stiffnesses become $M_r = 1$ and $K_r = \omega_r^2$, where ω_r represents the rth natural frequency of the elastica, in which case equation (47), upon expansion, becomes

$$\left\{\prod_{i=1}^{N}(\omega_i^2+\lambda^2)\right\}\left(\sum_{r=1}^{N}\frac{\phi_r^2(x_1)}{\omega_r^2+\lambda^2}+\frac{m\lambda^2+c\lambda+k}{(c\lambda+k)m\lambda^2}\right)\left(\sum_{r=1}^{N}\frac{\phi_r^2(x_2)}{\omega_r^2+\lambda^2}+\frac{1}{\lambda^2 M}\right)$$
$$-\left(\sum_{r=1}^{N}\frac{\phi_r(x_2)\phi_r(x_1)}{\omega_r^2+\lambda^2}\right)^2=0. \tag{48}$$

Comparing equations (48) and (18) of reference [10], notice the absence of the product terms. When the attachment location of the damped oscillator is not at a node of any normal mode of the linear elastic structure, then $\omega_i^2+\lambda^2 \neq 0$, for $i = 1, \dots, N$, and equation (48) reduces to equation (18) of reference [10]. Thus, using the proposed scheme, the frequency equation can be easily obtained, without the complexity associated with applying the Lagrange multipliers formalism outlined in reference [10].

The proposed approach also has an added benefit in that it allows all of the eigenvalues to be computed regardless where the attachment point is located. Because the Lagrange multipliers approach only leads to the characteristic determinant of equation (47) (see reference [10] for a detailed derivation), it "misses" some eigenvalues when the attachment point is at a node of any normal mode of the linear elastic structure. While this difficulty disappears if the linear elastica is artificially disassembled [2], additional work is required to recover these missing eigenvalues. The product terms that are present in equation (48) alleviate the aforementioned difficulty. In particular, the product terms signify that when the attachment point coincides with a node of any normal mode, then some of the natural frequencies of the combined system will be exactly the same as those of the linear structure. Thus, the proposed approach leads to a frequency equation that yields all of the eigenvalues of the system, even when the attachment point of the damped oscillator coincides with a node of any normal mode of the linear elastica.

4. CONCLUSIONS

An alternative formulation is proposed that can be used to determine the eigenvalues of a linear elastic structure carrying any number of concentrated masses, springs and viscous dampers. The proposed scheme leads to several noticeable advantages. Specifically, the proposed approach leads to a reduced characteristic determinant that is compact and simple to code; the method can be easily extended to accommodate any linear elastic structure with any boundary conditions; the scheme leads to a formulation such that the eigenvalues can be easily solved either graphically or numerically using any prepackaged code. Numerical experiments were performed to validate the proposed approach, and excellent agreements were found between the proposed scheme and known solutions published in the literature.

REFERENCES

1. E. H. DOWELL 1972 *Journal of Applied Mechanics* **39**, 727–732. Free vibrations of an arbitrary structure in terms of component modes.

2. E. H. DOWELL 1979 *Journal of Applied Mechanics* **46**, 206–209. On some general properties of combined dynamical systems.
3. M. GÜRGÖZE 1996 *Journal of Sound and Vibration* **190**, 149–162. On the eigenfrequencies of a cantilever beam with attached tip mass and a spring–mass system.
4. J. W. NICHOLSON and L. A. BERGMAN 1986 *Journal of Engineering Mechanics* **112**, 1–13. Free vibration of combined dynamical system.
5. G. G. G. LUESCHEN, L. A. BERGMAN and D. M. MCFARLAND 1996 *Journal of Sound and Vibration* **194**, 93–102. Green's functions for uniform Timoshenko beams.
6. S. KUKLA 1997 *Journal of Sound and Vibration* **205**, 355–363. Application of Green functions in frequency analysis of Timoshenko beams with oscillators.
7. L. ERCOLI and P. A. A. LAURA 1987 *Journal of Sound and Vibration* **114**, 519–533. Analytical and experimental investigation on continuous beams carrying elastically mounted masses.
8. P. D. CHA and C. PIERRE 1999 *Journal of Engineering Mechanics* **125**, 587–591. Frequency analysis of a linear elastic structure carrying a chain of oscillators.
9. P. D. CHA 2001 *Journal of Sound and Vibration* **247**, 185–194. Natural frequencies of a linear elastica carrying any number of sprung masses.
10. M. GÜRGÖZE and H. EROL 1999 *Journal of Sound and Vibration* **225**, 573–580. On the eigencharacteristics of longitudinally vibrating rods carrying a tip mass and viscously damped spring–mass in-span.
11. T. P. CHANG, F. I. CHANG and M. F. LIU 2001 *Journal of Sound and Vibration* **240**, 769–778. On the eigenvalues of a viscously damped simple beam carrying point masses and springs.
12. L. MEIROVITCH 1967 *Analytical Methods in Vibrations*. New York: The Macmillan Company.
13. R. F. BOISVERT, S. E. HOWE, D. K. KAHANER and J. L. SPRINGMANN 1990 *Guide to Available Mathematical Software*. Center for Computing and Applied Mathematics, National Institute of Standards and Technology, Gaithersburg, ML.
14. P. D. CHA and W. C. WONG 1999 *Journal of Sound and Vibration* **219**, 689–706. A novel approach to determine the frequency equations of combined dynamical systems.